

Exercises for Chapter 2: Linear Algebra

July 13, 2016

Exercises

1. The expression $\alpha \mathbf{u}$ for $\alpha \in \mathbb{R}$ and unit vector $\mathbf{u} \in \mathbb{R}^n$ defines a line of points that may be obtained by varying the value of α . Derive an expression for the point $\mathbf{y}$ that lies on this line that is as close as possible to an arbitrary point $\mathbf{x} \in \mathbb{R}^n$. This operation of replacing a point by its nearest member within some set is called *projection*.

Exercise contributed by Ian Goodfellow

Solutions

1. We begin by defining the distance from $\mathbf{y}$ to $\mathbf{x}$. We would like to find the $\mathbf{y}$ that minimizes this distance:

$$\|\mathbf{x} - \mathbf{y}\|^2. \quad (1)$$

Next, we need to enforce the constraint that $\mathbf{y}$ lies on the line defined by $\alpha \mathbf{u}$. We can do this simply by defining $\mathbf{y}$ to be $\alpha \mathbf{u}$.

$$\|\mathbf{x} - \alpha \mathbf{u}\|^2. \quad (2)$$

Next, we expand the expression:

$$\|\mathbf{x} - \alpha \mathbf{u}\|^2 \quad (3)$$

$$= (\mathbf{x} - \alpha \mathbf{u})^\top (\mathbf{x} - \alpha \mathbf{u}) \quad (4)$$

$$= \mathbf{x}^\top \mathbf{x} - 2\alpha \mathbf{x}^\top \mathbf{u} + \alpha^2 \mathbf{u}^\top \mathbf{u} \quad (5)$$

$$= \mathbf{x}^\top \mathbf{x} - 2\alpha \mathbf{x}^\top \mathbf{u} + \alpha^2. \quad (6)$$

In the last line, we used the fact that $\mathbf{u}$ is a unit vector to make the simplification $\mathbf{u}^\top \mathbf{u} = 1$.

We can minimize this distance by taking the derivative with respect to α and setting it to zero:

$$-2\mathbf{x}^\top \mathbf{u} + 2\alpha = 0 \quad (7)$$

$$\Rightarrow \alpha = \mathbf{x}^\top \mathbf{u}. \quad (8)$$

Recalling that $\mathbf{y} = \alpha \mathbf{u}$, we can conclude that $\mathbf{y} = \mathbf{x}^\top \mathbf{u} \mathbf{u}$.

Solution contributed by Ian Goodfellow